

# A FUNCTION WITH SUPPORT OF FINITE MEASURE AND “SMALL” SPECTRUM

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ABSTRACT. We construct a function on  $\mathbb{R}$  supported on a set of finite measure whose spectrum has density zero.

## 1. THE RESULT

Let  $F$  be a function in  $L^2(\mathbb{R})$ . We say that it is supported on  $S$  if

$$F = 0 \text{ almost everywhere on } \mathbb{R} \setminus S.$$

Suppose the set  $S \subset \mathbb{R}$  is of finite Lebesgue measure. Then the Fourier transform  $\widehat{F}$  of  $F$  is a continuous function, so the spectrum of  $F$  is naturally defined as the closure of the set where  $\widehat{F}$  takes non-zero values.

According to the uncertainty principle, the support and the spectrum of a (non-trivial) function  $F$  cannot be both “small sets”. This principle has various versions (see e.g. [HAJ94]).

In particular, the classic uniqueness theorem for analytic functions implies that if  $F$  is supported on an interval and it has a “spectral gap” (that is,  $\widehat{F} = 0$  on an interval) then  $F = 0$ .

Another important result says that if the support  $S$  and the spectrum  $Q$  of  $F$  are both of finite measure then  $F = 0$  [Ben74/85], [AB77].

On the other hand,  $F$  may have a support of finite measure and a spectral gap; see [Kr82], where such an example was constructed with  $F = 1_S$ .

Answering a question posed by Benedicks, Kargaev and Volberg [KV92] constructed an example of a function  $F$  such that

$$|S| < +\infty, |\mathbb{R} \setminus Q| = +\infty$$

(here and below by  $|A|$  we denote the Lebesgue measure of the set  $A$ ).

The goal of this note is to prove the following

**Theorem.** *There is a function  $F \in L^2(\mathbb{R})$  supported by a set  $S$  of finite measure, such that*

$$|Q \cap (-R, R)| = o(R) \text{ as } R \rightarrow \infty.$$

*In addition,  $F$  can be chosen as the indicator function of  $S$ .*

The proof below is based on a simple construction, completely different from the ones in the cited papers.

## 2. PROOF

2.1. Take a Schwartz function  $F_0$  such that

$$0 \leq F_0(t) \leq 1 \quad (t \in \mathbb{R})$$

and its Fourier transform  $\widehat{F}_0$  is positive on  $(-1, 1)$  and vanishes outside that interval. Define a sequence of functions  $F_n$  recursively by

$$F_n := F_{n-1} + G_n \quad (n = 1, 2, \dots),$$

where

$$(1) \quad G_n(t) := F_{n-1}(t)[1 - F_{n-1}(t)] \cos k_n t$$

We are going to prove that if the numbers  $k_n$  grow sufficiently fast, then the sequence  $F_n$  converges to a function  $F$  satisfying the requirements of the theorem.

2.2. Clearly,  $F_n$  and  $G_n$  are Schwartz functions.

A simple induction shows that for every  $t \in \mathbb{R}$ , we have

$$|G_n(t)| \leq \max\{F_{n-1}(t), 1 - F_{n-1}(t)\}$$

and

$$0 \leq F_n(t) \leq 1.$$

The Fourier transforms of  $F_{n-1}[1 - F_{n-1}]$ ,  $F_{n-1}^2[1 - F_{n-1}]$ , and  $F_{n-1}^2[1 - F_{n-1}]^2$  vanish outside a compact interval, so for each  $n \geq 1$ , we have:

$$\int_{\mathbb{R}} G_n = \int_{\mathbb{R}} F_{n-1} G_n = 0$$

and

$$\int_{\mathbb{R}} G_n^2 = \frac{1}{2} \int_{\mathbb{R}} F_{n-1}^2 [1 - F_{n-1}]^2,$$

provided that  $k_n$  is chosen sufficiently large. It follows that

$$(2) \quad \int_{\mathbb{R}} F_n = \int_{\mathbb{R}} F_0 =: C$$

and, thereby,

$$I_n := \int_{\mathbb{R}} F_n(1 - F_n) \leq C$$

(here, as usual, by  $C$  we denote a positive constant that may vary from line to line).

Observe also that

$$I_n = \int_{\mathbb{R}} [F_{n-1} + G_n][1 - F_{n-1} - G_n] = I_{n-1} - \int_{\mathbb{R}} G_n^2,$$

which implies that

$$\sum_{n \in [1, N]} \int_{\mathbb{R}} G_n^2 \leq I_0 - I_N \leq C,$$

and so

$$(3) \quad \sum_n \int_{\mathbb{R}} G_n^2 \leq C$$

2.3. Define the sequence  $Q_n$  of intervals on (another copy of)  $\mathbb{R}$  recursively as follows:

$$\begin{aligned} Q_0 &:= [-1, 1], \\ Q_n &:= \text{conv}(Q_{n-1} \cup [k_n + 2Q_{n-1}] \cup [-k_n + 2Q_{n-1}]) \end{aligned}$$

(here  $\text{conv } E$  denotes the convex hull of a set  $E \subset \mathbb{R}$ ). Clearly, for every  $n$ ,

$$\begin{aligned} \text{spec } F_{n-1} &\subset Q_{n-1}; \\ \text{spec } G_n &\subset [k_n + 2Q_{n-1}] \cup [-k_n + 2Q_{n-1}]. \end{aligned}$$

Set  $Q := Q_0 \cup \bigcup_n ([k_n + 2Q_{n-1}] \cup [-k_n + 2Q_{n-1}])$ .

Choosing  $k_n$  growing sufficiently fast we can ensure that the spectra of  $G_n$  are pairwise disjoint and

$$(4) \quad |Q \cap (-R, R)| = o(R) \text{ as } R \rightarrow \infty.$$

2.4. Consider the series  $F_0 + G_1 + G_2 + \dots$ . Since the spectra of the terms are pairwise disjoint, this series is orthogonal in  $L^2(\mathbb{R})$ . Then (3) implies that it converges in  $L^2(\mathbb{R})$  to some non-trivial function  $F$ . The partial sums of this series are  $F_n$ . Take a subsequence  $F_{n_\ell}$  such that

$$F_{n_\ell} \rightarrow F \text{ almost everywhere on } \mathbb{R} \text{ as } \ell \rightarrow \infty.$$

Recall that all  $F_n$  are non-negative functions, so (2) implies that

$$(5) \quad F \geq 0 \text{ almost everywhere and } \int_{\mathbb{R}} F < \infty.$$

It follows from (1) and (3) that

$$\sum_n \int_{\mathbb{R}} [F_n(1 - F_n)]^2 = 2 \sum_n \int_{\mathbb{R}} G_n^2 < +\infty,$$

so we must have

$$F(1 - F) = \lim_{\ell \rightarrow \infty} F_{n_\ell}(1 - F_{n_\ell}) = 0 \text{ almost everywhere,}$$

which implies that  $F$  is the indicator-function of a set  $S$ . According to (5), this set has finite measure. Clearly the spectrum of  $F$  is a subset of  $Q$ . Due to (4) it has density zero. This finishes the proof.

**Remark.** *Consider the function*

$$h(R) := |Q \cap (-R, R)|.$$

*In the conditions of the Theorem, it can not be bounded. However the proof above shows that it may increase arbitrarily slowly. It remains an open question, however, if  $Q$  can have uniform density 0, i.e., if it is possible that*

$$\lim_{R \rightarrow \infty} \sup_{x \in \mathbb{R}} \frac{1}{2R} |Q \cap (x - R, x + R)| = 0.$$

#### REFERENCES

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